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Extension of the N=2 Virasoro algebra by two primary fields of dimension 2 and 3

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Abstract

We explicitly construct the extension of the $N = 2$ super Virasoro algebra by two super primary fields of dimension two and three with vanishing $u(1)$-charge. Using a super covariant formalism we obtain two different solutions both consistent for generic values of the central charge c. The first one can be identified with the super $\mathcal{W}_4$-algebra - the symmetry algebra of the CP(3) Kazama-Suzuki model. With the help of unitarity arguments we predict the self-coupling constant of the field of dimension two for all super $\mathcal{W}_n$-algebras. The second solution is special in the sense that it does not have a finite classical limit $c \to \infty$ and generic null fields appear. In the spirit of recent results in the $N = 0$ case it can be understood as a unifying $N = 2$ super $\mathcal{W}$-algebra for all CP(n) coset models. It does not admit any unitary representation.

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1 Introduction

In 1989, motivated by the work of Gepner [11], Kazama and Suzuki [20,21] were looking for $N = 2$ theories having minimal and unitary representations for $c = 9$. They constructed and classified coset models of $N = 1$ Kac-Moody algebras which have an extended $N = 2$ supersymmetry. The symmetry algebras of these models should be generically existing $N = 2$ super $\mathcal{W}$-algebras [15,23]. Recently, another occurrence of $N = 2$ super $\mathcal{W}$-algebras was established by Bershadsky et al. [4], realizing that the ghost-sector of $\mathcal{W}_n$-gravity theories coupled to matter contains $N = 2$ super $\mathcal{W}$-algebras.
Ito [15,16] found via quantum hamiltonian reduction based on the affine super Lie algebra $\mathcal{A}(n, n-1)$ that the symmetry algebra of the CP$(n-1)$ Kazama-Suzuki model should be the super $\mathcal{W}_n$-algebra, an extension of the $N = 2$ super Virasoro algebra by bosonic superfields of dimension 2 up to $n-1$. The explicit form of the classical super $\mathcal{W}_3$-algebra has been found by Lu et al. [22] via Polyakov construction. Later, using conformal bootstrap techniques, Romans [26] explicitly constructed its quantum version which has been the only generically existing nonlinear quantum $N = 2$ super $\mathcal{W}$-algebra constructed up to now. This algebra was investigated further, classical [19] and quantum [1,17,18] free field realizations were found. Also higher classical super extensions of the $\mathcal{W}_3$-algebra [30] were constructed, like the $N = 4$ super $\mathcal{W}_3$-algebra [25] by the 'dual formalism' [24]. Inami et al. [14] constructed the extension of the $N = 2$ super Virasoro algebra by a superfield of dimension $\frac{3}{2}$. In [6] extensions by superfields of dimension higher than two were investigated systematically, yielding only non-deformable solutions.
Our motivation in this letter was to extend the super $\mathcal{W}_n$-series. Besides the expected super $\mathcal{W}_4$ solution we found a second one which can be regarded as a 'unifying' [7,8] super $\mathcal{W}$-algebra for the CP(n) Kazama-Suzuki models.
This letter is organized as follows: In the second section we review an $N = 2$ supersymmetric manifestly covariant formalism. In the third section we give the algorithm for construction of the super $\mathcal{W}_4$-algebra finding two solutions to be consistent for generic value of the central charge c. In the fourth section we present and discuss the two solutions giving a short report on unifying algebras. In the fifth section we conclude this letter presenting a short summary.

2 Holomorphic N=2 supersymmetric CFT

We present some formulae about the general structure of holomorphic $N = 2$ superconformal field theories [6,27]: using the definitions

$$Z_{12} = z_1 - z_2 - \tfrac{\theta_1^+\theta_2^- + \theta_1^-\theta_2^+}{2}\, , \theta_{12}^\pm = \theta_1^\pm - \theta_2^\pm\, , D^\pm = \partial_{\theta^\pm} + \tfrac{1}{2}\theta^\mp\partial_z \tag{2.1}$$

the $N = 2$ super Virasoro field is given by:

$$\mathcal{L}(Z) = J(z) + \tfrac{\theta^+G^-(z) - \theta^-G^+(z)}{\sqrt{2}} + \theta^+\theta^-L(z). \tag{2.2}$$

An $N = 2$ holomorphic superfield of super conformal dimension Δ and $u(1)$-charge Q is denoted by:

$$\Phi_\Delta^Q(Z) = \varphi(z) + \tfrac{\theta^+\psi^-(z) - \theta^-\psi^+(z)}{\sqrt{2}} + \theta^+\theta^-\chi(z) \equiv \Phi_1(z) + \tfrac{\theta^+\Phi_2(z) - \theta^-\Phi_3(z)}{\sqrt{2}} + \theta^+\theta^-\Phi_4(z). \tag{2.3}$$

The two-point function reads:

$$\langle \Phi^{Q_i}_{\Delta_i}(Z_1)\Phi^{Q_j}_{\Delta_j}(Z_2)\rangle = \frac{D_{ij}}{Z_{12}^{2\Delta_i}}\left(1-\frac{Q_i}{2}\frac{\theta^+_{12}\theta^-_{12}}{Z_{12}}\right)\delta_{\Delta_i,\Delta_j}\delta_{Q_i,-Q_j} \tag{2.4}$$

with some constant D_{ij}.
The three-point function is more complicated since several cases have to be distinguished:

(i) If $Q_i + Q_j = Q_k$ then

$$\langle \Phi^{-Q_k}_{\Delta_k}(Z_1)\,\Phi^{Q_i}_{\Delta_i}(Z_2)\,\Phi^{Q_j}_{\Delta_j}(Z_3)\rangle = C_{ijk}\frac{2+\gamma_1\alpha_{ijk}\Theta^+_{ijk}\Theta^-_{ijk}}{Z_{12}^{\gamma_1}\,Z_{23}^{\gamma_2}\,Z_{13}^{\gamma_3}}\prod_{m<n}(1+\tfrac{Q_n-Q_m}{6}\tfrac{\theta^+_{mn}\theta^-_{mn}}{Z_{mn}}). \tag{2.5}$$

(ii) If $Q_i + Q_j = Q_k \pm 1$ then

$$\langle \Phi^{-Q_k}_{\Delta_k}(Z_1)\,\Phi^{Q_i}_{\Delta_i}(Z_2)\,\Phi^{Q_j}_{\Delta_j}(Z_3)\rangle = \frac{C_{ijk}\Theta^{\pm}_{ijk}}{Z_{12}^{\gamma_1}\,Z_{23}^{\gamma_2}\,Z_{13}^{\gamma_3}}(1+\textstyle\sum_{m<n}\tfrac{Q_n-Q_m}{6}\tfrac{\theta^+_{mn}\theta^-_{mn}}{Z_{mn}}) \tag{2.6}$$

where $\gamma_1 = \Delta_k + \Delta_i - \Delta_j$, $\gamma_2 = \Delta_i + \Delta_j - \Delta_k$ and $\gamma_3 = \Delta_k + \Delta_j - \Delta_i$. C_{ijk} and α_{ijk} are *independent* parameters.

$$\Theta^{\pm}_{ijk} = \frac{Z_{ij}\theta^{\pm}_k + Z_{jk}\theta^{\pm}_i + Z_{ki}\theta^{\pm}_j + \frac{1}{6}\left(\theta^{\pm}_i\theta^{\mp}_{jk}(\theta^{\pm}_j+\theta^{\pm}_k)+\theta^{\pm}_j\theta^{\mp}_{ki}(\theta^{\pm}_k+\theta^{\pm}_i)+\theta^{\pm}_k\theta^{\mp}_{ij}(\theta^{\pm}_i+\theta^{\pm}_j)\right)}{\sqrt{Z_{ij}Z_{jk}Z_{ki}}} \tag{2.7}$$

denotes the pair of Grassmann odd $N=2$ super Möbius ($\equiv Osp(2,2)$) invariants [27].
Following [6], the coupling constants C_{ij}^k occurring in the OPE are determined by the linear system $C_{ij}^l D_{lk} = C_{ijk}$.
For our calculations we use a different but equivalent form of the three-point function which is not invariant under permutations.
Naive normal ordered products (NOPs), defined naturally by terms of the regular part of the OPE, are not quasiprimary, but one can project them onto their quasiprimary component $\mathcal{N}_s$, e.g.

$$\mathcal{N}_s(\mathcal{L}\mathcal{L}) = N_s(\mathcal{L}\mathcal{L}) - \tfrac{1}{3}[D^-,D^+]\mathcal{L}. \tag{2.8}$$

3 Algorithm

We will apply the manifestly covariant formalism presented in [6] to the construction of an $N=2$ super $\mathcal{W}$-algebra with *three* generators, $\mathcal{SW}(1,2,3)$: We extend the $N=2$ super Virasoro algebra by two additional super primary fields $\mathcal{W}(\Delta=2, Q=0)$ and $\mathcal{V}(\Delta=3, Q=0)$. Note that it would be almost impossible to directly construct an $N=0$ $\mathcal{W}$-algebra containing twelve generators. Only the $N=2$ structure makes such a calculation practicable. We choose the following normalization of the two-point functions:

$$\langle \mathcal{W}(Z_1)\mathcal{W}(Z_2)\rangle = \tfrac{c/2}{Z_{12}^4}\,, \qquad \langle \mathcal{V}(Z_1)\mathcal{V}(Z_2)\rangle = \tfrac{c/3}{Z_{12}^6}. \tag{3.1}$$

For explicit calculations we expand the components of the superfields into Fourier modes, making it possible to apply Lie algebra methods such as the universal polynomials [5].

We use $Mathematica^{TM}$[29] and the C-program *'commute'* [12]. For consistency of the algebra all commutators have to satisfy Jacobi identities leading to restrictions on the self-couplings and the central charge c. Thus we proceed as follows:

- We write down the most general ansätze for the super OPEs schematically given by:

$$\begin{aligned} \mathcal{W} \circ \mathcal{W} &= [\mathcal{L}] + [\mathcal{W}] + [\mathcal{V}] \\ \mathcal{V} \circ \mathcal{W} &= [\mathcal{L}] + [\mathcal{W}] + [\mathcal{V}] + [\mathcal{W}\mathcal{W}] \\ \mathcal{V} \circ \mathcal{V} &= [\mathcal{L}] + [\mathcal{W}] + [\mathcal{V}] + [\mathcal{W}\mathcal{W}] + [\mathcal{V}\mathcal{W}]. \end{aligned} \tag{3.2}$$

For each dimension occurring in the OPE we need a basis of super quasiprimary normal ordered products. All relevant NOPs are shown in table 1.

Δ	Q	NOPs $\mathcal{F}$
1	0	$\mathcal{L}$
2	0	$\mathcal{N}_s(\mathcal{L}\mathcal{L})\,;\mathcal{W}$
3	0	$\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{L}[D^-,D^+]\mathcal{L})\,;\mathcal{N}_s(\mathcal{W}\mathcal{L})\,;\mathcal{V}$
$\frac{7}{2}$	∓ 1	$\mathcal{N}_s(\mathcal{L}D^\pm\partial\mathcal{L})\,;\mathcal{N}_s(\mathcal{W}D^\pm\mathcal{L})$
4	0	$\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\mathcal{L})\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}[D^-,D^+]\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{L}\partial^2\mathcal{L})\,;$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{W}[D^-,D^+]\mathcal{L})\,,\mathcal{N}_s(\mathcal{W}\partial\mathcal{L})\,;\mathcal{N}_s(\mathcal{V}\mathcal{L})\,;\mathcal{N}_s(\mathcal{W}\mathcal{W})$
$\frac{9}{2}$	∓ 1	$\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}D^\pm\partial\mathcal{L})\mathcal{L})\,;\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}D^\pm\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{W}D^\pm\partial\mathcal{L})\,;\mathcal{N}_s(\mathcal{V}D^\pm\mathcal{L})$
5	0	$\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\mathcal{L})\mathcal{L})\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}[D^-,D^+]\mathcal{L})\mathcal{L})\mathcal{L})\,,$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\partial^2\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{L}[D^-,D^+]\partial^2\mathcal{L})\,,\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}D^\pm\partial\mathcal{L})D^\mp\mathcal{L})\,;$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}\mathcal{L})\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}[D^-,D^+]\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{W}[D^-,D^+]\partial\mathcal{L})\,,$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}D^\pm\mathcal{L})D^\mp\mathcal{L})\,,\mathcal{N}_s(\mathcal{W}\partial^2\mathcal{L})\,;\mathcal{N}_s(\mathcal{N}_s(\mathcal{V}\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{V}[D^-,D^+]\mathcal{L})\,,$
		$\mathcal{N}_s(\mathcal{V}\partial\mathcal{L})\,;\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}\mathcal{W})\mathcal{L})\,,\mathcal{N}_s(\mathcal{W}[D^-,D^+]\mathcal{W})\,;\mathcal{N}_s(\mathcal{V}\mathcal{W})$
$\frac{11}{2}$	∓ 1	$\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\mathcal{L})\mathcal{L})\mathcal{L})D^\pm\mathcal{L})\,,\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}[D^-,D^+]\mathcal{L})\mathcal{L})D^\pm\mathcal{L})\,,$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}D^\pm\partial\mathcal{L})\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\partial^2\mathcal{L})D^\pm\mathcal{L})\,;\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}\partial\mathcal{L})D^\pm\mathcal{L})\,,$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}D^\pm\mathcal{L})\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}[D^-,D^+]\mathcal{L})D^\pm\mathcal{L})\,,\mathcal{N}_s(\mathcal{W}D^\pm\partial^2\mathcal{L})\,,$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}D^\pm\partial\mathcal{L})\mathcal{L})\,;\mathcal{N}_s(\mathcal{N}_s(\mathcal{V}D^\pm\mathcal{L})\mathcal{L})\,,\mathcal{N}_s(\mathcal{V}D^\pm\partial\mathcal{L})\,;\mathcal{N}_s(\mathcal{W}D^\pm\partial\mathcal{W})\,,$
		$\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}\mathcal{W})D^\pm\mathcal{L})\,;\mathcal{N}_s(\mathcal{V}D^\pm\mathcal{W})$

Table 1: Quasiprimary NOPs up to dimension $\frac{11}{2}$

The Kac determinants in the vacuum sector are presented in table 2.

Δ	$\det(D_\Delta) \sim$
1	c
2	$c^2(c-1)$
3	$c^4(c-1)(c+6)(2c-3)(5c-12)$
$\frac{7}{2}$	$c^4(c-1)^2(c+3)^2$
$\frac{9}{2}$	$c^8(c-1)^4(c+3)^2(c+6)^2(2c-3)^2(5c-12)^2$

Table 2: Kac determinants

We omit the levels 4, 5 and $\frac{11}{2}$ since these determinants contain a priori unknown self-couplings. They will be tabulated below after having presented the solutions obtained from Jacobi identities.
Although three-point functions involving $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{F}$ vanish identically for all fields $\mathcal{F} \in [\mathcal{L}]$, it is necessary to assume the appearance of $[\mathcal{L}]$ in the OPE $\mathcal{V} \circ \mathcal{W}$ because the D-matrices (D_{ij}) are not blockdiagonal with respect to the conformal families, whenever $[\mathcal{W}\mathcal{W}]$ or $[\mathcal{V}\mathcal{W}]$ appear.

• The next step is to expand the fields into their components and to evaluate the structure constants for the lowest components. We obtain the super coupling constants by which all structure constants for the higher components are determined [6]. Note that $N = 2$ supersymmetry implies that we only have to calculate one D-matrix for every super-conformal dimension and one or two coupling constants per field (there are about 30 structure constants of the components!). In this step covariance reduces the effort considerably.

• Finally, we have to check Jacobi identities. It is sufficient to check them only for the additional primaries because Jacobi identities involving the super Virasoro field are satisfied automatically by $Osp(2,2)$-invariance. Let $(\phi_i\phi_j\phi_k\phi_l)$ be the Jacobi identity equivalent to the associativity of the four point function involving these four fields. Exploiting symmetries such as charge conjugation, there are only eight types of Jacobi identities for the component fields of one superfield with $u(1)$-charge $Q = 0$:

$$(\varphi\varphi\varphi\varphi)\,,(\varphi\varphi\varphi\chi)\,,(\varphi\varphi\chi\chi)\,,(\varphi\chi\chi\chi)\,,(\chi\chi\chi\chi)\,,(\varphi\varphi\psi^+\psi^-)\,,(\psi^+\psi^-\chi\chi)\,,(\psi^+\psi^-\psi^+\psi^-). \tag{3.3}$$

Altogether we have to check 48 Jacobi identities, but only four of them turn out to be independent.

4 Results

In this section we present the results of our calculations. We note beforehand that we found two and only two solutions existing for generic value of c which we denote by $\mathcal{SW}(1,2,3)^{[\mathrm{I,II}]}$. The OPE $\mathcal{W} \circ \mathcal{W}$ is given in table 3.

NOP $\mathcal{F}$	$C_{\mathcal{W}\mathcal{W}}^{\mathcal{F}}$	$C_{\mathcal{W}\mathcal{W}}^{\mathcal{F}}\alpha_{\mathcal{W}\mathcal{W}\mathcal{F}}$
$\mathcal{L}$	0	3
$\mathcal{N}_s(\mathcal{L}\mathcal{L})$	$-\frac{3}{2(c-1)}$	0
$\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\mathcal{L})\mathcal{L})$	0	$-\frac{3(4c+3)}{(2c-3)(c-1)(c+6)}$
$\mathcal{N}_s(\mathcal{L}[D^-,D^+]\mathcal{L})$	0	$\frac{23c-30}{2(2c-3)(c+6)}$
$\mathcal{N}_s(\mathcal{L}D^\pm\partial\mathcal{L})$	$-\frac{6}{c-1}$	$-$
$\mathcal{V}$	0	$\frac{3}{2}C_{\mathcal{V}\mathcal{W}}^{\mathcal{W}}\alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}$
$\mathcal{W}$	$C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}$	0
$\mathcal{N}_s(\mathcal{W}\mathcal{L})$	0	$\frac{14C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}}{5c-12}$
$\mathcal{N}_s(\mathcal{W}D^\pm\mathcal{L})$	$\pm\frac{12C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}}{c+3}$	$-$

Table 3: Structure constants of the OPE $\mathcal{W} \circ \mathcal{W}$

We skip the presentation of the other structure constants since they are complicated rational functions in c and the couplings among the primaries themselves. They simplify only considerably after inserting the results obtained from checking Jacobi identities. This also fixes the values of the self-couplings appearing in table 3. A complete list of structure constants can be found in [28].

4.1 $\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$, the super $\mathcal{W}_4$-algebra

We obtain a first solution for which the four free couplings in the ansatz (3.2) are:

$$
\begin{aligned}
\left(C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}\right)^2 &= \frac{(c+3)^2(7c-27)^2}{2(1-c)(c-21)(c+9)(5c-9)} \\
\left(C_{\mathcal{V}\mathcal{W}}^{\mathcal{W}}\alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}\right)^2 &= \frac{36(c-33)(c-3)(c-1)^2(c+3)(c+18)}{(c-21)(c+6)(c+9)(2c-3)(5c-12)(5c-9)} \\
\left(C_{\mathcal{V}\mathcal{W}}^{\mathcal{V}}\right)^2 &= \frac{9(c+6)^2(2c-3)^2(7c-27)^2}{2(9-5c)(c-21)(c-1)(c+9)(5c-12)^2} \\
\left(C_{\mathcal{V}\mathcal{V}}^{\mathcal{V}}\alpha_{\mathcal{V}\mathcal{V}\mathcal{V}}\right)^2 &= \\
&\frac{4\left(1312200-1234926c+393093c^2-72306c^3+7641c^4+123c^5+7c^6\right)^2}{(c-33)(c-21)(c-3)(c+3)(c+6)(c+9)(c+18)(2c-3)(5c-12)^3(5c-9)}.
\end{aligned}
\tag{4.1.1}
$$

The remaining Kac determinants now read as in table 4.

Δ	$\det(D_\Delta)\sim$
4	$c^8(c-33)(c-3)(c-1)^5(c+3)(c+6)(c+18)(2c-3)(5c-12)(5c+27)(7c-27)/((c-21)(c+9))$
5	$c^{18}(c-57)(c-33)(c-18)(c-3)^2(c-1)^{11}(c+3)^4(c+6)^4(c+15)(c+18)^2(c+36)(2c-9)(2c-3)^4(5c-12)^4(5c+27)^2(7c-27)^2/\left((c-21)^3(c+9)^3(5c-9)^2\right)$
$\frac{11}{2}$	$c^{28}(c-33)^2(c-27)^2(c-18)^2(c-3)^2(c-1)^{20}(c+3)^{10}(c+6)^6(c+15)^2(c+18)^2(2c-3)^6(5c-12)^6(5c+27)^4(7c-27)^2/\left((c-21)^6(c+9)^6(5c-9)^4\right)$

Table 4: Kac determinants [I]

In contrast to the second solution this algebra has a finite classical limit $c\to\infty$ which is a hint that only this solution can be the super $\mathcal{W}_4$-algebra. In order to support this assumption, we use the decomposition of CP(n) Kazama-Suzuki models into three bosonic coset models (see [26] and references therein):

$$
\frac{\widehat{su}(n+1)_k\oplus\widehat{so}(2n)_1}{\widehat{su}(n)_{k+1}\oplus\widehat{u}(1)} \cong \frac{\widehat{su}(k)_n\oplus\widehat{su}(k)_1}{\widehat{su}(k)_{n+1}} \oplus \frac{\widehat{su}(n)_k\oplus\widehat{su}(n)_1}{\widehat{su}(n)_{k+1}} \oplus \widehat{u}(1). \tag{4.1.2}
$$

The equivalence has to be understood as up to questions of finite reducibility of the representation theory [10].

Using the T-equivalence [2,9]

$$\frac{\widehat{su}(k)_n \oplus \widehat{su}(k)_1}{\widehat{su}(k)_{n+1}} \cong \frac{\widehat{su}(n+1)_k}{\widehat{su}(n)_k \oplus \widehat{u}(1)} \tag{4.1.3}$$

we can substitute the first coset on the r.h.s. of (4.1.2) by a bosonic CP(n) coset. For $n = 1$ one can rigorously prove a dual equivalence [2].
Suppose $\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$ to be the symmetry algebra of the CP(3) Kazama-Suzuki model. Then the decomposition (4.1.2) will induce a transformation of the subalgebra of the fields of dimension two, namely $N(JJ), L, \varphi = \mathcal{W}_1$ into a direct sum of three Virasoro algebras. These three Virasoro fields can be constructed explicitly: Denoting

$$\gamma = \sqrt{4c - \left(C_{\mathcal{WW}}^{\mathcal{W}}\right)^2 + c\left(C_{\mathcal{WW}}^{\mathcal{W}}\right)^2} \tag{4.1.4}$$

we obtain three mutually commuting Virasoro generators with their corresponding central charges:

- bosonic CP(n)-part

$$T_1 = \tfrac{\gamma + \sqrt{c-1}C_{\mathcal{WW}}^{\mathcal{W}}}{2\gamma} L - \tfrac{\sqrt{c-1}}{\gamma}\varphi - \tfrac{3\left(\gamma + \sqrt{c-1}C_{\mathcal{WW}}^{\mathcal{W}}\right)}{4c\gamma} N(JJ) \tag{4.1.4a}$$

$$\tilde{c}_1 = (c-1)\frac{\gamma + \sqrt{c-1}C_{\mathcal{WW}}^{\mathcal{W}}}{2\gamma},$$

- $\mathcal{W}_n$-part

$$T_2 = \tfrac{\gamma - \sqrt{c-1}C_{\mathcal{WW}}^{\mathcal{W}}}{2\gamma} L + \tfrac{\sqrt{c-1}}{\gamma}\varphi - \tfrac{3\left(\gamma - \sqrt{c-1}C_{\mathcal{WW}}^{\mathcal{W}}\right)}{4c\gamma} N(JJ) \tag{4.1.4b}$$

$$\tilde{c}_2 = (c-1)\frac{\gamma - \sqrt{c-1}C_{\mathcal{WW}}^{\mathcal{W}}}{2\gamma},$$

- $\widehat{u}(1)$-part

$$T_3 = \tfrac{3}{2c} N(JJ) \tag{4.1.4c}$$

$$\tilde{c}_3 = 1.$$

Of course, the self-coupling $C_{\mathcal{WW}}^{\mathcal{W}}$ depends on the super $\mathcal{W}$-algebra. Inserting the solution (4.1.1) yields the central charges:

$$\begin{aligned} \tilde{c}_1 &= \frac{(c+9)(5c-9)}{3(c+27)} \\ \tilde{c}_2 &= \frac{2c(21-c)}{3(c+27)} \\ \tilde{c}_3 &= 1. \end{aligned} \tag{4.1.5}$$

Unitary representations of $\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$ are also unitary representations of the $\mathcal{W}_3$-part. Therefore,

$$\tilde{c}_2 = 2\left(1 - \frac{12}{(k+3)(k+4)}\right) \tag{4.1.6}$$

is a necessary condition for c of the unitary series of $\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$. Plugging (4.1.6) in (4.1.5) we get a quadratic equation. Its solution consists of one ascending and one descending branch of central charges:

$$c = \frac{9k}{k+4}, \qquad c = \frac{9(k+7)}{(k+7)-4}. \tag{4.1.7}$$

The first branch is actually the unitary series of the CP(3) Kazama-Suzuki model, confirming the identification of $\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$ as its symmetry algebra - the super $\mathcal{W}_4$-algebra. The second series can be formally obtained from the first one by the substitution $k \to -k-7$. It is related to non-compact coset models (see ref. [3] for more details).
Furthermore, we have looked for primary fields in the different summands of (4.1.2). Actually, in the $\mathcal{W}_3$-part there is a spin-3 primary with respect to T_2 generating a $\mathcal{W}(2,3)$ together with T_2 - we have checked that there is no spin-4 generator in this part. Concerning the investigation of the bosonic CP(n) models in ref. [8] there should be a $\mathcal{W}(2,3,\ldots,19)$ in the CP(3)-part - we constructed the fields of dimension 3 and 4 explicitly. Those three fields are presented in the appendix (A.1-A.3).
The formulae (4.1.3-4.1.4) do not depend on the special structure of $\mathcal{SW}(1,2,3)$. Reversing the whole procedure one can obtain the general self-coupling constant of the field of dimension two for any super $\mathcal{W}_{n+1}$-algebra. To this end, we require c and $\tilde{c}_2$ to take values in the unitary series of the CP(n) Kazama-Suzuki model and the $\mathcal{W}_n$ model, respectively:

$$c = \frac{3nk}{k+n+1}, \qquad \tilde{c}_2 = (n-1)\left(1 - \frac{n(n+1)}{(k+n)(k+n+1)}\right) \tag{4.1.8}$$

Inserting $\tilde{c}_2$ in (4.1.4b) and eliminating k by c yields:

$$\left(C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}\right)^2 = \frac{(c+3)^2\left((2n+1)c - 3n^2\right)^2}{(n-1)\,(c-1)\,(c+3n)\,(6n+3-c)\,((n+2)c-3n)}. \tag{4.1.9}$$

In the following we will use this formula for the identification of the second solution $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$.

4.2 $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$, a unifying super $\mathcal{W}$-algebra

In order to identify the second solution, we summarize the most important results about $N=0$ unifying $\mathcal{W}$-algebras [7,8].
Unifying $\mathcal{W}$-algebras interpolate the rank n of Casimir algebras $\mathcal{WL}_n$ at particular values of the central charge. Using the coset realizations of these algebras, the unifying $\mathcal{W}$-algebras can be regarded as a generalization of level-rank-duality [2,9]. These identifications between a priori different $\mathcal{W}$-algebras for particular values of the central charge are closely related to the fact that for certain values of the central charge some generators become null fields leading to a 'truncation' of the $\mathcal{W}$-algebra. Another important feature of unifying algebras is that certain fields which one would expect to appear in the singular part of the OPE of the primaries are null fields. Finally, the absence of a finite classical limit $c \to \infty$ is characteristic for unifying $\mathcal{W}$-algebras in all explicitly known cases.

The structure constants of the second solution among the primaries are:

$$\begin{aligned}
\left(C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}\right)^2 &= \frac{c(c+3)^2}{2(1-c)(1+c)} \\
\left(C_{\mathcal{V}\mathcal{W}}^{\mathcal{W}}\alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}\right)^2 &= \frac{8(24-c)(c-1)^2(c+3)^2}{3(c+1)(c+6)(2c-3)(5c-12)} \\
\left(C_{\mathcal{V}\mathcal{W}}^{\mathcal{V}}\right)^2 &= \frac{9c(c+6)^2(2c-3)^2}{2(1-c)(1+c)(5c-12)^2} \\
\left(C_{\mathcal{V}\mathcal{V}}^{\mathcal{V}}\alpha_{\mathcal{V}\mathcal{V}\mathcal{V}}\right)^2 &= \frac{(24-c)\left(1296-2484c+36c^2+315c^3+74c^4\right)^2}{6(c+1)(c+3)^2(c+6)(2c-3)(5c-12)^3}.
\end{aligned} \tag{4.2.1}$$

Again the rest of the Kac determinants becomes quite simple (see table 5).

Δ	$\det(D_\Delta) \sim$
4	$c^9(c-1)^5(c+3)^2(c+6)(2c-3)(5c-12)(5c+27)$
5	$c^{20}(c-1)^{11}(c+3)^6(c+6)^4(c+8)^2(2c-3)^4(5c-12)^4(5c+27)^2/(c+1)^2$
$\frac{11}{2}$	$c^{28}(c-1)^{18}(c+3)^{14}(c+6)^4(c+8)^2(2c-3)^4(5c-12)^4(5c+27)^2/(c+1)^2$

Table 5: Kac determinants [II]

There is a pair of generic null fields of dimension $\frac{11}{2}$ whose explicit form can be found in the appendix (A.4). For that reason the determinant at this level is based on only 26 fields, not on 28 as it is for $\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$.
The occurrence of those null fields and the non-existing classical limit $c \to \infty$ of the structure constants give rise to the conjecture that this algebra is a unifying super $\mathcal{W}$-algebra.
Inserting the solution (4.2.1) in (4.1.4a-c), the central charge simply splits in the following way:

$$c = -2 + (c+1) + 1, \tag{4.2.2}$$

corresponding to the decomposition:

$$\frac{\widehat{su}(n+1)_{-\frac{n}{2}} \oplus \widehat{so}(2n)_1}{\widehat{su}(n)_{-\frac{n-2}{2}} \oplus \widehat{u}(1)} \cong \frac{\widehat{su}(n+1)_{-\frac{n}{2}}}{\widehat{su}(n)_{-\frac{n}{2}} \oplus \widehat{u}(1)} \oplus \frac{\widehat{su}(n)_{-\frac{n}{2}} \oplus \widehat{su}(n)_1}{\widehat{su}(n)_{-\frac{n-2}{2}}} \oplus \widehat{u}(1) \tag{4.2.3}$$

which we obtain from (4.1.2) by inserting (4.1.3) and formally substituting k by $-\frac{n}{2}$. Note that the bosonic CP(n) model at level $k = -\frac{n}{2}$ always carries the central charge $\tilde{c} = -2$ [8].
The same substitution applied to the unitary series of the CP(3) Kazama-Suzuki model should lead to a truncation of $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$ to the super $\mathcal{W}_{n+1}$-algebra for $n < 3$ and vice versa for $n > 3$ (for $n = 3$ one has $\mathcal{SW}(1,2,3)^{[\mathrm{II}]} = \mathcal{SW}(1,2,3)^{[\mathrm{I}]}$) at

$$c = -\frac{3n^2}{n+2}. \tag{4.2.4}$$

This can be shown explicitly for all structure constants of the first three examples:

- $c = -1$: All self-coupling constants diverge. After rescaling the fields with a factor $(c+1)^x$ such that the structure constants will be finite, $\mathcal{W}$ and $\mathcal{V}$ become null fields.
- $c = -3$: $C_{\mathcal{V}\mathcal{W}}^{\mathcal{W}} \alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}$ and $C_{\mathcal{V}\mathcal{V}}^{\mathcal{V}} \alpha_{\mathcal{V}\mathcal{V}\mathcal{V}}$ diverge. The redefined field $\mathcal{V}$ will be a null field and $C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}$ is equal to the self-coupling constant of the super $\mathcal{W}_3$-algebra [26].
- $c = -\frac{27}{5}$: The structure constants of $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$ are equal to the corresponding ones of $\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$.

That reads briefly as in table 6.

c	truncation
-1	$\mathcal{SW}(1,2,3)^{[\mathrm{II}]} \to SVir_{N=2}$
-3	$\mathcal{SW}(1,2,3)^{[\mathrm{II}]} \to \mathcal{SW}(1,2)$
$-\frac{27}{5}$	$\mathcal{SW}(1,2,3)^{[\mathrm{II}]} = \mathcal{SW}(1,2,3)^{[\mathrm{I}]}$

Table 6: Truncations

Furthermore, for $n > 3$ $C_{\mathcal{W}\mathcal{W}}^{\mathcal{W}}$ of $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$ is equal to the structure constant (4.1.9) of the super $\mathcal{W}_{n+1}$-algebra for the corresponding value of c, confirming the identification of $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$ as a unifying super $\mathcal{W}$-algebra.
Finally, we have checked that the symmetry algebras of the first and second part in the decomposition (4.2.3) are a $\mathcal{W}(2,3)$ with internal $\tilde{c} = -2$ and the Virasoro algebra, respectively. The spin-3 field is shown in the appendix (A.5).
In virtue of $\tilde{c} = -2$ in the $\mathcal{W}(2,3)$ symmetry algebra there is no unitary representation of $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$ for any value of c: Suppose $\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$ to have a unitary representation. Due to the meaning of equivalence in (4.1.2) it will be decomposable into unitary representations of the symmetry algebras of the cosets on the r.h.s. of (4.2.3). However, the $\mathcal{W}(2,3)$ symmetry algebra will not admit unitary representations because of its negative central charge.

5 Conclusion and outlook

Using a manifestly covariant formalism, we were able to determine the complete structure of $\mathcal{SW}(1,2,3)$. The most important result is the existence of *two* generic solutions.
The first one can be identified with the super $\mathcal{W}_4$-algebra possessing a finite classical limit $c \to \infty$. Using a decomposition of the underlying coset model - on the algebraic level one obtains three subalgebras - and inserting their well-known unitary series we recovered the ascending unitary branch of the CP(3) Kazama-Suzuki model. There is a descending second series due to its non-compact version. Furthermore, we identified the subalgebras and constructed some of their generators. Inverting this way of argumentation leads to a prediction of the self-coupling constant of the field of dimension two for every super $\mathcal{W}_n$-algebra.
The second solution has no finite classical limit $c \to \infty$. The decomposition mentioned above leads to subalgebras with few generators: besides a $\widehat{u}(1)$-part a Virasoro algebra and a $\mathcal{W}(2,3)$ with internal $\tilde{c} = -2$ independent of the overall value of c. Furthermore, the occurrence of a pair of generic null fields strongly indicates that this solution is a unifying $N = 2$ super $\mathcal{W}$-algebra for the CP(n) Kazama-Suzuki models at negative central charges.

For the first three examples we checked this explicitly on the level of structure constants. In the case $n > 3$ at least the self-coupling constants of the field of dimension two coincide using the prediction mentioned above. Finally, we pointed out that this algebra does not admit any unitary representation.

We would like to end with several open questions which quite naturally arise given our results:

(i) Do other $\mathcal{SW}(1,2,...,n)$ algebras also admit solutions different from the super $\mathcal{W}_{n+1}$-algebra? (cf. ref. [13] in the $N=0$ case)

(ii) What are the unifying super $\mathcal{W}$-algebras for other Kazama-Suzuki models?

(iii) Are there unifying super $\mathcal{W}$-algebras admitting unitary representations?

(iv) Do Kazama-Suzuki models exhaust the classification of $N=2$ unitary models?

(v) Are there non-unitary minimal $N=2$ models?

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Appendix

$\mathcal{SW}(1,2,3)^{[\mathrm{I}]}$:

• the spin-3 generator of $\mathcal{W}(2,3)$ for $\tilde{c} = \frac{4(21-c)c}{c+27}$:

$$\sqrt{\frac{2(33-c)}{5c+27}}\frac{6c(c-21)}{(c-1)(c+6)(c+27)(2c-3)}\Big(\frac{2(c-1)(c+6)(c+9)(2c-3)(5c-9)C_{\mathcal{WW}}^{\mathcal{W}}}{3c(c+3)(5c-12)(7c-27)}N(\mathcal{W}_1 J)$$
$$+\frac{(c+6)(c+9)(2c-3)(5c-9)C_{\mathcal{VW}}^{\mathcal{W}}\alpha_{\mathcal{VWW}}}{18c(c-33)}\mathcal{V}_1 - \frac{(c-1)(c+6)(c+9)(2c-3)(5c-9)C_{\mathcal{WW}}^{\mathcal{W}}}{9(c+3)(5c-12)(7c-27)}\mathcal{W}_4$$
$$+\frac{c}{18}\partial^2 J + \frac{c}{6}\partial L - \frac{c}{6}N(G^- G^+) + N(LJ) - \frac{1}{c}N(N(JJ)J)\Big) \tag{A.1}$$

• the spin-3 generator of the CP(3) part for $\tilde{c} = \frac{2(c+9)(5c-9)}{c+27}$:

$$\sqrt{\frac{6(3-c)(c+18)}{c(c-27)}}\frac{3(c+9)(5c-9)}{(c-1)(c+6)(c+27)(2c-3)}\Big(\frac{4(c-21)(c-1)(c+6)(2c-3)C_{\mathcal{WW}}^{\mathcal{W}}}{3(c+3)(5c-12)(7c-27)}N(\mathcal{W}_1 J)$$
$$+\frac{c(c-21)(c+6)(2c-3)C_{\mathcal{VW}}^{\mathcal{W}}\alpha_{\mathcal{VWW}}}{54(c-3)(c+18)}\mathcal{V}_1 - \frac{2c(c-21)(c-1)(c+6)(2c-3)C_{\mathcal{WW}}^{\mathcal{W}}}{9(c+3)(5c-12)(7c-27)}\mathcal{W}_4$$
$$+\frac{c}{18}\partial^2 J + \frac{c}{6}\partial L - \frac{c}{6}N(G^- G^+) + N(LJ) - \frac{1}{c}N(N(JJ)J)\Big) \tag{A.2}$$

• the spin-4 generator of the CP(3) part for $\tilde{c} = \frac{2(c+9)(5c-9)}{c+27}$:

$$\sqrt{\frac{54(27-7c)(c-21)(c+3)(c+9)(5c-9)}{(c-81)(c-27)(c-3)(c-1)(c+18)(c+27)(5c+27)}}\Big(\frac{c(c+9)(1458-909c+84c^2+7c^3)C_{\mathcal{WW}}^{\mathcal{W}}}{72(3-2c)(c+3)^2(c+6)(7c-27)^2}\partial^3 J$$
$$+\frac{-1944+1557c+373c^2}{c(c-1)(5c-12)(1377+246c+25c^2)}N(N(\mathcal{W}_1 J)J) - \frac{2(2430-4059c+1350c^2+223c^3)}{3(c+3)(c-1)(5c-12)(1377+246c+25c^2)}N(\mathcal{W}_1 L)$$
$$+\frac{c(c+9)(1458-909c+84c^2+7c^3)C_{\mathcal{WW}}^{\mathcal{W}}}{12(c+3)^2(c+6)(2c-3)(7c-27)^2}N(\partial G^- G^+) - \frac{(c-21)(c+9)(5c-9)C_{\mathcal{WW}}^{\mathcal{W}}C_{\mathcal{VW}}^{\mathcal{W}}\alpha_{\mathcal{VWW}}}{6(c+3)^2(7c-27)^2}N(\mathcal{V}_1 J)$$
$$+\frac{(c+9)(1458-909c+84c^2+7c^3)C_{\mathcal{WW}}^{\mathcal{W}}}{2(c+3)^2(c+6)(2c-3)(7c-27)^2}N(N(G^- J)G^+) + \frac{(c+9)(1458-909c+84c^2+7c^3)C_{\mathcal{WW}}^{\mathcal{W}}}{2(3-2c)(c+3)^2(c+6)(7c-27)^2}N(\partial L J)$$
$$+\frac{(c-6)(c+9)(1458-909c+84c^2+7c^3)C_{\mathcal{WW}}^{\mathcal{W}}}{12(3-2c)(c+3)^2(c+6)(7c-27)^2}N(G^- \partial G^+) + \frac{c(c-21)(c+9)(5c-9)C_{\mathcal{WW}}^{\mathcal{W}}C_{\mathcal{VW}}^{\mathcal{W}}\alpha_{\mathcal{VWW}}}{54(c+3)^2(7c-27)^2}\mathcal{V}_4$$
$$-\frac{(c-21)(c+9)(5c-9)(-135+140c+3c^2)C_{\mathcal{WW}}^{\mathcal{W}}}{3(c+3)^2(7c-27)^2(1377+246c+25c^2)}N(\mathcal{W}_1\mathcal{W}_1) - \frac{2916-2457c+96c^2+29c^3}{12(c+3)(5c-12)(1377+246c+25c^2)}\partial^2\mathcal{W}_1$$

$$-\frac{(c+9)(-3542940+2589408c+1216053c^2+265545c^3-552258c^4+1194c^5+4937c^6+141c^7)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{4c(c-1)(c+3)^2(c+6)(2c-3)(7c-27)^2(1377+246c+25c^2)}N(N(JJ)L)$$
$$+\frac{(c+9)(590490-1496637c+1241676c^2-352494c^3+13866c^4+859c^5)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{6(c-1)(c+3)(c+6)(2c-3)(7c-27)^2(1377+246c+25c^2)}N(LL)-\frac{2}{(c+3)(5c-12)}N(\mathcal{W}_4J)$$
$$+\frac{3(c+9)(-1062882+1200663c-59049c^2-13257c^3-74907c^4+6846c^5+346c^6)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{2c^2(c-1)(c+3)^2(c+6)(2c-3)(7c-27)^2(1377+246c+25c^2)}N(N(JJ)N(JJ))$$
$$+\frac{3(c+9)(-1653372+685260c+122445c^2-1422c^3-17088c^4+770c^5+47c^6)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{4c(c+3)^2(c+6)(2c-3)(7c-27)^2(1377+246c+25c^2)}N(N(LJ)J)$$
$$-\frac{(c+9)(2125764-3254256c+1198719c^2+108432c^3-87174c^4+1848c^5+107c^6)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{8c(c+3)^2(c+6)(2c-3)(7c-27)^2(1377+246c+25c^2)}N(\partial J\partial J)$$
$$+\frac{(c+9)(-1062882+2630961c-1045872c^2-89964c^3+47376c^4+987c^5+34c^6)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{(1377+246c+25c^2)12(3-2c)(c+3)^2(c+6)(7c-27)^2}\partial^2 L$$
$$-\frac{c}{6(c+3)(5c-12)}N(\mathcal{W}_3G^-)+\frac{c}{6(c+3)(5c-12)}N(\mathcal{W}_2G^+)\Big) \tag{A.3}$$

$\mathcal{SW}(1,2,3)^{[\mathrm{II}]}$:

• the pair $\Phi^\pm$ of generic null fields of dimension $\frac{11}{2}$:

$$\Phi^\pm\sim\pm\frac{36+180c+293c^2-8c^3-4c^4}{8c(c-1)(c+1)(c+6)(2c-3)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\mathcal{L})\mathcal{L})\mathcal{L})D^\mp\mathcal{L})$$
$$\pm\frac{288-810c+3919c^2-2379c^3-52c^4}{84c(c-1)(c+1)(c+6)(2c-3)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}[D^-,D^+]\mathcal{L})\mathcal{L})D^\mp\mathcal{L})$$
$$+\frac{33-4c}{2(c-1)(2c-3)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}D^\mp\partial\mathcal{L})\mathcal{L})\mathcal{L})\pm\frac{-108+185c+38c^2}{42c(1-c)(c+1)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{L}\partial^2\mathcal{L})D^\mp\mathcal{L})$$
$$+\frac{2(144-234c+41c^2)C^{\mathcal{W}}_{\mathcal{V}\mathcal{W}}\alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}}{7c(c-24)(c-1)}\mathcal{N}_s(\mathcal{V}D^\mp\partial\mathcal{L})\pm\frac{(c+1)(5c-12)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}C^{\mathcal{W}}_{\mathcal{V}\mathcal{W}}\alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}}{c(24-c)(c+3)}\mathcal{N}_s(\mathcal{V}D^\mp\mathcal{W})$$
$$\pm\frac{12(5c-12)C^{\mathcal{W}}_{\mathcal{V}\mathcal{W}}\alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}}{5c(c-24)(c-1)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{V}D^\mp\mathcal{L})\mathcal{L})\pm\frac{18(4c+3)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{5c(3-2c)(c+3)(c+6)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}D^\mp\mathcal{L})\mathcal{L})\mathcal{L})$$
$$+\frac{3(-648-1440c+939c^2+71c^3)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{5c(3-2c)(c+3)(c+6)(5c-12)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}\partial\mathcal{L})D^\mp\mathcal{L})\pm\frac{1}{c}\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}\mathcal{W})D^\mp\mathcal{L})$$
$$\pm\frac{(3672+3852c-4254c^2+114c^3+95c^4)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{5c(c+3)(c+6)(2c-3)(5c-12)}\mathcal{N}_s(\mathcal{W}D^\mp\partial^2\mathcal{L})+\mathcal{N}_s(\mathcal{W}D^\mp\partial\mathcal{W})$$
$$\pm\frac{7(-216+144c-93c^2+53c^3)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{5c(c+3)(c+6)(2c-3)(5c-12)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}[D^-,D^+]\mathcal{L})D^\mp\mathcal{L})$$
$$+\frac{48(-324+81c+138c^2+7c^3)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{25c(c+3)(c+6)(2c-3)(5c-12)}\mathcal{N}_s(\mathcal{N}_s(\mathcal{W}D^\mp\partial\mathcal{L})\mathcal{L}) \tag{A.4}$$

• the spin-3 generator of $\mathcal{W}(2,3)$ for $\tilde{c}=-2$:

$$\sqrt{\frac{6(c-24)}{c}}\frac{6}{(c-1)(c+6)(2c-3)}\Big(\frac{2(c-1)(c+1)(c+6)(2c-3)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{3c(c+3)(5c-12)}N(\mathcal{W}_1J)$$
$$+\frac{(c+1)(3-2c)(c+6)C^{\mathcal{W}}_{\mathcal{V}\mathcal{W}}\alpha_{\mathcal{V}\mathcal{W}\mathcal{W}}}{6(c-24)}\mathcal{V}_1+\frac{(1-c)(c+1)(c+6)(2c-3)C^{\mathcal{W}}_{\mathcal{W}\mathcal{W}}}{9(c+3)(5c-12)}\mathcal{W}_4$$
$$+\frac{c}{18}\partial^2 J+\frac{c}{6}\partial L-\frac{c}{6}N(G^-G^+)+N(LJ)-\frac{1}{c}N(N(JJ)J)\Big) \tag{A.5}$$

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